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# POLYNOMIAL SOLUTIONS IN PROBLEMS OF CONTROLLING THE HEATING OF SOLIDS

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*We consider an approximate method of solving the problem of optimum control of the process that is described by a homogeneous heat-conduction equation. The governing parameter enters the boundary condition, while the integral quadratic expression is subjected to minimization. A solution of the problem is found for a class of controls described by a polynomial of finite degree. An example is considered.*

One of the trends in solving the problem of reducing the energy consumed in manufacturing products is the industrial implementation of regimes of heating solids that ensure the smallest expenditures of thermal energy with a high quality of the materials and items produced. The processes of heat treatment of concrete and ferroconcrete to accelerate solidification at contemporary enterprises of the building industry require the expenditure of energy resources that greatly exceeds the theoretically possible ones to ensure the necessary strength of items; therefore, of current interest is the search for heating régimes that would be satisfactory as regards heat expenditures.

The analytical relations that make it possible to determine the optimum values of the temperature of a heating medium during the heating of concrete and ferroconcrete items can be obtained from the solution of the system of differential equations that model the process of heat treatment.

Let the controlled process be described by the function $\theta(x, F)$ that in the region $Q = (0 < x < 1, 0 < F \leq T)$ satisfies the heat-conduction equation

$$\frac{\partial \theta}{\partial F} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\Gamma}{x} \frac{\partial \theta}{\partial x}, \quad \Gamma = 0, 1, 2, \tag{1}$$

and at the boundary $Q$ satisfies the conditions

$$\theta (x, 0) = \frac{\partial \theta (0, F)}{\partial x} = 0 ; \tag{2}$$

$$\frac{\partial \theta (1, F)}{\partial x} = \mathrm{Bi} (\theta_{\mathrm{m}} (F) - \theta (1, F)) . \tag{3}$$

It is required to find the governing function $\theta_{\mathrm{m}}(F)$ for which the functional

$$J (\theta_{\mathrm{m}}) = \int_0^1 (\theta (x, T) - \varphi (x))^2 x^{\Gamma} dx + \beta \int_0^T \theta_{\mathrm{m}}^2 (F) dF , \tag{4}$$

where $\varphi(x)$ is the given function and $\beta > 0$, takes on the least value [1].

Problem (1)-(3) at $\Gamma = 0$ was considered in [1], where the solution in the class of piecewise-constant functions is given. In [2, 3], the solution of this problem is found in the class $L_2$, and the means of construct-

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ing approximate solutions is given. We note that the theoretical justification of the method in [2, 3] involves a fair number of difficulties.

We will seek an approximate solution of the posed problem in the form of the polynomial

$$\theta_m(F) = q_1 F + q_2 F^2 + \ldots + q_m F^m , \quad \theta_m(0) = 0 . \tag{5}$$

The basis for approximation (5) is the fact that any function from the class $L_2$ can be represented, with any degree of accuracy, by a polynomial [4].

We take from [5] the solution of problem (1)-(3):

$$\theta(x, F) = \sum_{n=1}^{\infty} r_n \frac{\Phi_\Gamma(\mu_n x)}{V_\Gamma(\mu_n)} \int_0^F \mu_n \theta_m(F_1) \exp(\mu_n^2 (F_1 - F))\, dF_1 =$$

$$= \sum_{k=1}^{m} q_k \sum_{n=1}^{\infty} r_n \mu_n \frac{\Phi_\Gamma(\mu_n x)}{V_\Gamma(\mu_n)} a_{nk}(F) , \tag{6}$$

where $\Phi_\Gamma(\xi)$ and $V_\Gamma(\xi)$ are the functions defined in [5]; $\mu_n$ are the roots of the characteristic equation $\Phi_\Gamma(\mu)\mathrm{Bi} = V_\Gamma(\mu)\mu$;

$$r_n = \frac{2\,\mathrm{Bi}^2}{\mu_n^2 + \mathrm{Bi}^2 + (1 - \Gamma)\,\mathrm{Bi}} ; \quad a_{nk}(F) = \exp(-\mu_n^2 F) \int_0^F F_1^k \exp(\mu_n^2 F_1)\, dF_1 . \tag{7}$$

Substituting expressions (5) and (6) into Eq. (4), we find

$$J(\theta_m) = \sum_{k,l=0}^{m} A_{kl} q_k q_l + 2 \sum_{k=1}^{m} B_k q_k + B_0 . \tag{8}$$

The values of the coefficients $A_{kl} = A_{lk}$, $B_k$, and $B_0$ are found from the following expressions:

$$A_{kl} = \int_0^1 x^\Gamma \left( \sum_{n=1}^{\infty} r_n \mu_n \frac{\Phi_\Gamma(\mu_n x)}{V_\Gamma(\mu_n)} a_{nk}(T) \right) \left( \sum_{j=1}^{\infty} r_j \mu_j \frac{\Phi_\Gamma(\mu_j x)}{V_\Gamma(\mu_j)} a_{jl}(T) \right) dx +$$

$$+ \beta \int_0^T F^{k+l}\, dt = \sum_{n,j=1}^{\infty} \frac{r_n \mu_n r_j \mu_j}{V_\Gamma(\mu_n) V_\Gamma(\mu_j)} a_{nk}(T)\, a_{jl}(T) \int_0^1 x^\Gamma \Phi_\Gamma(\mu_n x)\, \Phi_\Gamma(\mu_j x)\, dx + \frac{T^{k+l+1}}{k + l + 1} \beta .$$

Since

$$\int_0^1 x^\Gamma \Phi_\Gamma(\mu_n x)\, \Phi_\Gamma(\mu_j x)\, dx = \begin{cases} 0 \ \text{ for } j \neq n , \\ \dfrac{V_\Gamma^2(\mu_n)}{r_n} \ \text{ for } j = n , \end{cases}$$

we have

$$A_{kl} = \sum_{n=1}^{\infty} r_n \mu_n^2 a_{nk}(T)\, a_{nl}(T) + \frac{T^{k+l+1}}{k + l + 1} \beta ; \tag{9}$$

$$B_k = -\sum_{n=1}^{\infty} \frac{r_n \mu_n}{V_\Gamma(\mu_n)} a_{nk}(T)\, \varphi_n , \tag{10}$$

where $\Phi_n = \int_0^1 x^\Gamma \varphi(x) \Phi_\Gamma(\mu_n x) dx$ (for $\varphi(x) \equiv 1$ $\varphi_n = \frac{V_\Gamma(\mu_n)}{\mu_n} 0$) are the Fourier coefficients of the function $\varphi(x)$ in the expansion in terms of the system $\{\Phi_\Gamma(\mu_n x)\}$;

$$B_0 = \int_0^1 x^\Gamma \varphi^2(x)\, dx . \tag{11}$$

Thus, the functional $J(\theta_m)$ becomes a quadratic function of the variables $q_1, q_2, ..., q_m$. The extremum values of the coefficients $q_1, ..., q_m$ can be found from the system of equations

$$\frac{1}{2}\frac{\partial J}{\partial q_k} = \sum_{k=1}^{m} A_{kl}\, q_l + B_k = 0 , \quad k = 1, 2, ..., m . \tag{12}$$

To investigate system (12), we will introduce the linear set $\{l\}$ of convergent sequences:

$$l = (P(F), a_1, a_2, ...) ,$$

where $F \in [0; T]$; $0 = (0, 0, ...)$; $P(F)$ are the finite-degree polynomials.

For any two elements from $\{l\}$

$$l_1 = (P_1(F), a_1^{(1)}, a_2^{(1)}, ...) , \quad l_2 = (P_2(F), a_1^{(2)}, a_2^{(2)}, ...)$$

we introduce the binary operation

$$(l_1; l_2) = \beta \int_0^T P_1(F)\, P_2(F)\, dF + \sum_{n=1}^{\infty} r_n \mu_n^2 a_n^{(1)} a_n^{(2)} .$$

It can easily be verified that the operation $(l_1; l_2)$ satisfies all the conditions of determining the scalar product [4].

Note further that the sequences

$$a_k = (F^k, a_{1k}(T), a_{2k}(T), ...) , \quad k = 1, 2, ..., m ,$$

belong to $\{l\}$ and form a linearly independent system (this follows already from the fact that the set of the degrees $F, F^2, ..., F^m$ is linearly independent). Expressions (9) show that $A_{k,l} = (a_k; a_l)$, i.e., the matrix

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{m1} & \cdots & A_{mm} \end{pmatrix},$$

which is the Gram matrix of the linearly independent system $a_k$ ($k = 1, ..., m$), is positive definite, $\det A > 0$. From this it follows that system (12) has only one solution:

$$\overset{0}{q}_1, \overset{0}{q}_2, ..., \overset{0}{q}_m . \tag{13}$$

Moreover, the second differential

$$d^2 J = \sum_{k,l=1}^{m} A_{kl} h_k h_l$$

is the quadratic form with the positive-definite matrix $A$, i.e., $d^2 J > 0$, and solution (13) is the point of the minimum of the functional $J(\theta_m)$.

The proposed method makes it possible to seek the optimum regimes of heat treatment of solids, selecting the rise in temperature by the linear, quadratic, or any other polynomial law. The necessary calculations can be made in each specific case with the aid of a computer.

To illustrate the proposed method, we will consider an example from [3]:

$$\frac{\partial \theta}{\partial F} = \frac{\partial^2 \theta}{\partial x^2} ; \quad \theta (x, 0) = \left. \frac{\partial \theta}{\partial x} \right|_{x=0} = 0 ;$$

$$\left. \frac{1}{\mathrm{Bi}} \frac{\partial \theta}{\partial x} \right|_{x=1} = \theta_m (F) - \theta (1, F) ; \quad J (\theta_m) = \int_0^1 (\theta (x, T) - 1)^2 \, dx + \int_0^T \theta_m^2 (F) \, dF ,$$

where $0 \le \theta_m(F) \le 1$, $T = 10$, and $\mathrm{Bi} = 0.5$.

To attain an accuracy of $10^{-6}$, the number of terms in series (6), (9), and (10) must be $N \ge 30$. In the work we selected $N = 100$.

The coefficients $a_{nk}$ are calculated following the recurrence formula

$$a_{n1} = \frac{T}{\mu_n^2} + \frac{1}{\mu_n^4} (\exp (- \mu_n^2 T) - 1) ; \quad a_{nk} = \frac{1}{\mu_n^2} (T^k - k a_{n,k-1}) .$$

Instead of the coefficients $q_k$ we will seek $P_k = q_k T^k$, $k = 1, 2, ..., m$. Here $a_{nk}$ should be replaced by $\tilde{a}_{nk}$, defined by the formulas

$$\tilde{a}_{n1} = \frac{1}{\mu_n^2} - \frac{1}{T \mu_n^4} (\exp (- \mu_n^2 T) - 1) ; \quad \tilde{a}_{nk} = \frac{1}{\mu_n^2} \left( 1 - \frac{k}{T} \tilde{a}_{n,k-1} \right),$$

and the value of the functional $J(\theta_m)$ will be written in the form

$$J (\theta_m) = 1 - B_1 P_1 - B_2 P_2 - ... - B_m P_m .$$

Performing the calculations, we obtain for $m \le 5$

$A_{11} = 3.92635$, $A_{12} = 2.99397$, $A_{13} = 2.42559$, $A_{14} = 2.04153$, $A_{15} = 1.76405$;

$A_{22} = 2.41157$, $A_{23} = 2.02135$, $A_{24} = 1.74106$, $A_{25} = 1.52972$;

$A_{33} = 1.73430$, $A_{34} = 1.51942$, $A_{35} = 1.35234$;

$A_{44} = 1.34858$, $A_{45} = 1.21266$;

$A_{55} = 1.09958$;

$B_1 = 0.76994$, $B_2 = 0.64113$, $B_3 = 0.55222$, $B_4 = 0.48627$, $B_5 = 0.43506$.

Solving system (12) that corresponds to the considered problem, we find

$m = 2$ : $J(\theta_m = 0.82631$, $P_1 = -0.12436$, $P_2 = 0.42025$;

$m = 4$ : $J(\theta_m) = 0.82494$, $P_1 = -0.13853$, $P_2 = 0.42974$, $P_3 = 0.01124$;

$m = 4$ : $J(\theta_m) = 0.82315$, $P_1 = 0.09364$, $P_2 = -0.12405$, $P_3 = -0.00039$, $P_4 = 0.37944$;

$m = 5 : J(\theta_m) = 0.82313$, $P_1 = 0.05292$, $P_2 = 0.02591$, $P_3 = 0.00001$, $P_4 = 0$, $P_5 = 0.27470$.

The values of $J(\theta_m)$ at $m = 4$ and $m = 5$ coincide with the values found in [3].

## NOTATION

$x$, $F$, $\theta$, and $\theta_m$, coordinate, time (Fourier number), temperature of the body, and temperature of the heating medium (nondimensional); Bi, Biot number; $J(\theta_m)$, minimized functional.

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